

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You can achieve 50 points in total.

1. [5+5 Points.]

Consider the growth model

$$x' = x - h(1 + \sin t),$$

where x denote the size of a population and $-h(1 + \sin t)$ is a periodic harvesting term.

- (a) Determine the general solution of this system, and show that there is exactly one periodic solution. What is the condition on h , and what is the biological interpretation of this condition?
- (b) Compute the Poincaré map for this system, and use it to verify your result from part (a) that there is exactly one periodic solution.

2. [4+6 Points.]

Consider the pair of two-dimensional systems

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X \text{ and } Y' = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} Y.$$

- (a) Determine the phase portraits of the two systems.
- (b) Find a topological conjugacy between the two systems.

3. [9+6 Points.]

- (a) For each of the following bifurcations, give an example of a family of one-dimensional systems of the form $x' = f_a(x)$ where $a \in \mathbb{R}$ is a parameter such that at $a = 0$ one finds
 - i. a transcritical bifurcation,
 - ii. a saddle-node bifurcation, and
 - iii. a pitchfork bifurcation.

Also sketch the corresponding bifurcation diagrams.

(b) Consider a first-order differential equation

$$x' = f_a(x)$$

for which $f_a(x_0) = 0$ and $f'_a(x_0) \neq 0$. Prove that the differential equation

$$x' = f_{a+\epsilon}(x)$$

has an equilibrium point $x_0(\epsilon)$ where $\epsilon \mapsto x_0(\epsilon)$ is a smooth function satisfying $x_0(0) = x_0$ for ϵ sufficiently small.

4. [10+5 Points.]

(a) Prove that the equilibrium at the origin $(x, y, z) = 0$ of the system

$$\begin{aligned}x' &= -x^3 \\y' &= -y(x^2 + z^2 + 1) \\z' &= -\sin z\end{aligned}$$

is asymptotically stable and determine its basin of attraction.

(b) State Lasalle's Invariance Principle.